

One-Loop Amplitudes in $\mathcal{N} = 4$ Super Yang-Mills and Anomalous Dual Conformal Symmetry

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Abstract

We discuss what predictions can be made for one-loop superamplitudes in maximally supersymmetric Yang-Mills theory by using anomalous dual conformal symmetry. We show that the anomaly coefficient is a specific combination of two-mass hard and one-mass supercoefficients which appears in the supersymmetric on-shell recursion relations and equals the corresponding tree-level superamplitude. We discuss further novel relations among supercoefficients imposed by the remaining non-anomalous part of the symmetry. In particular, we find that all one-loop supercoefficients, except the four-mass box coefficients, can be expressed as linear combinations of three-mass box coefficients and a particular symmetric combination of two-mass hard coefficients. We check that our equations are explicitly satisfied in the case of one-loop n -point MHV and NMHV amplitudes. As a bonus, we prove the covariance of the NMHV superamplitudes at an arbitrary number of points, extending previous results at $n \leq 9$.

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1 Introduction

Recently, a new symmetry of planar scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills (SYM) has been proposed in [1], called dual superconformal symmetry. This symmetry is expected to be exact at tree level, and violated by an anomaly at the quantum level. Indeed, it was proved in [2] that the tree-level S -matrix of the planar $\mathcal{N} = 4$ theory is covariant under the dual superconformal symmetry, and similar covariance properties were also established for the supercoefficients of the one-loop expansion of the scattering amplitudes in a basis of box functions [2, 3].

On the string theory side, the origin of this symmetry has been explained neatly in [4, 5] using a T-duality of the superstring theory on $AdS_5 \times S^5$, which involves a bosonic T-duality [6] and fermionic T-duality. A certain combination of these bosonic and fermionic T-dualities maps the original string sigma model into a dual sigma model which turns out to be identical to the original one. Moreover, the T-duality exchanges the original with the dual superconformal symmetries.

In contrast to the ordinary superconformal symmetry of the maximally supersymmetric theory, its dual counterpart is not a symmetry of the action but only of the amplitudes at the planar level. Interestingly, the authors of [7] considered the commutation relations between ordinary and dual superconformal generators and discovered that these give rise to a Yangian symmetry at tree level. Constraints imposed on amplitudes by the ordinary superconformal symmetry have recently been considered in [8].

An important feature of the strong coupling calculation of scattering amplitudes in the dual sigma model is that it is identical to that of a Wilson loop with a special polygonal contour, constructed by gluing together the lightlike momenta of the scattered particles following the order of the insertions of the string vertex operators [6]. Strikingly, there is now growing evidence that calculations in weakly coupled $\mathcal{N} = 4$ SYM of the same Wilson loops – specifically at one [9, 10] and two loops [11–14] – are in perfect agreement with the MHV scattering amplitudes of the $\mathcal{N} = 4$ theory calculated in [15–18]. A numerical calculation of Wilson loops at two loops for an arbitrary number of edges, or scattered particles, has been carried out in [19], and awaits explicit results for the corresponding amplitudes in $\mathcal{N} = 4$ SYM.

At weak coupling, the origin of the dual conformal symmetry was understood from the Wilson loop perspective in [12], where it was noted that it is nothing but the ordinary conformal symmetry of the Wilson loop acting on the 't Hooft's region (or T-dual) momenta x_i 's, defined via

$$p_{i,\alpha\dot{\alpha}} = (x_i - x_{i+1})_{\alpha\dot{\alpha}} , \quad (1.1)$$

where $i = 1, \dots, n$. Here n is the number of scattered particles, and we identify $x_{n+1} = x_1$. Importantly, due to the presence of cusps in the polygonal contour, the symmetry is anomalous. In [11], an expression for the Wilson loop anomaly was proposed at one loop, and later extended and proved to all loops in [12]. Furthermore, the four- and five-point Wilson loops are completely determined (up to a constant) by the anomalous dual conformal Ward identity [12], and found to be of the form predicted by the ABDK/BDS ansatz for the amplitudes [18, 20].

Based on the connection between Wilson loops and MHV amplitudes, an all-loop expression of the anomaly of generic loop amplitudes was proposed in [1]. Schematically, this anomaly is proportional to the corresponding tree-level amplitude, multiplied by the all-loop anomaly of the Wilson loop.

In this paper we wish to investigate the origin of this anomaly on the amplitude side, and use the expression of the anomaly conjectured in [1] to make predictions for the supercoefficients of generic (in particular also non-MHV) one-loop superamplitudes. Specifically, our strategy will consist in applying the dual conformal generators to a generic superamplitude, which can be written [15] as a linear combination of box functions times supercoefficients. We will then use the covariance of the one-loop supercoefficients proved in [2] and [1], which allows the conformal generators to pass through the supercoefficients and thus act on the box functions.

Now, box functions are (with the exception of the four-mass box) infrared divergent, and need to be regularised, which is usually accomplished by evaluating them in $4 - 2\epsilon$ dimensions. However, if one could work in four dimensions (by e.g. resorting to a “regularisation” of the box which makes the external kinematics massive), the boxes would be invariant under the dual conformal symmetry [21] – a fact that has been referred to as “pseudo-conformality” of these integral functions, and has played an important role in higher-loop calculations of $\mathcal{N} = 4$ MHV amplitudes [16, 22–27]. Dimensional regularisation cleanly exposes the anomalies of the box functions. We will therefore calculate all these box anomalies, and use them in order to write down the general expression for the anomaly of an arbitrary non-MHV amplitude. This turns out to be proportional to a particular linear combination of supercoefficients.

The equations we derive exhibit two important features. Firstly, the one-loop anomaly conjectured in [1] arises naturally in our setup as the particular combination of two-mass hard and one-mass coefficients considered in [29], which is equal to the tree-level amplitude. Furthermore, we find a new set of relations which need to be satisfied if the expression of the all-loop anomaly of [1] is correct. By assuming that this is indeed the case, we derive new equations which we use to relate supercoefficients of one-loop superamplitudes. Specifically, we find that we can re-express all supercoefficients (except the four-mass ones) in terms of three-mass box coefficients and a particular symmetric combination of two-mass hard coefficients. Four-mass

coefficients multiply integrals which are finite [28] and dual conformal invariant [21], hence we are not able to constrain them.

Finally, we will present some checks of the new equations we propose. Specifically, we will consider the infinite sequences of MHV and NMHV superamplitudes in $\mathcal{N} = 4$ SYM computed in [15] and [3], respectively, and prove that their expressions satisfy our conformal equations. The proof of dual conformal invariance in the NMHV case requires for $n \geq 10$ particles a new identity for the superconformal invariant R_{rst} in terms of which the NMHV superamplitude was expressed in [3]. We show this in an Appendix to this paper. A further brief Appendix is devoted to constructing combinations of box functions which are dual conformal invariant.

The rest of the paper is organised as follows.¹ In the next Section, after writing the expansion of generic amplitudes in terms of box functions to review our notation, we will calculate the anomalies of all types of one-loop box functions. We will explain how this anomaly arises from an anomalous conformal Jacobian, which formally vanishes as $\epsilon \rightarrow 0$, but leads to a nonvanishing contribution because of the presence of infrared divergences. We will use these results to calculate, in Section 3, the expression of the dual conformal anomaly of an arbitrary n -point one-loop superamplitude in terms of the supercoefficients. In Section 4 we analyse the conformal equations. Firstly, we show that these equations include, as a subset, the infrared consistency conditions, and we present a way to disentangle the new equations from the infrared equations. Then, we show how these equations can in principle be solved in terms of three-mass box coefficients and a symmetric combination of two-mass hard coefficients. Finally, we check in Section 5 that our equations are explicitly satisfied in the case of one-loop n -point MHV and NMHV amplitudes.

Note added: After finishing this work, we were made aware in recent email correspondence with Henriette Elvang and Dan Freedman of a work of Elvang, Freedman and Kiermaier [31] which has some overlap with our paper.

2 One-loop amplitudes and anomalous box functions

In this section we start off by introducing the expansion of generic n -point superamplitudes in terms of box functions. We will then move on to derive the dual conformal anomalies of the different box functions, i.e. zero-mass, one-mass, two-mass easy, two-

¹The results of this paper were announced at the “International Workshop on Gauge and String Amplitudes” at Durham University, 30 March–3 April 2009 [30].

mass hard, and three-mass. The four-mass box is conformal invariant, and we will not have anything to say about the supercoefficients of these functions.

2.1 One-loop superamplitudes

In the supersymmetric formalism of [32], to each particle in the $\mathcal{N} = 4$ theory one associates commuting spinors $\lambda_\alpha, \tilde{\lambda}_{\dot{\alpha}}$ (in terms of which the momentum of the i^{th} particle is $p_{\alpha\dot{\alpha}}^i = \lambda_\alpha^i \tilde{\lambda}_{\dot{\alpha}}^i$), as well as anticommuting variables η_i^A , where $A = 1, \dots, 4$ is an $SU(4)$ index. The supersymmetric amplitude can then be expanded in powers of the $\mathcal{N} = 4$ superspace coordinates η_A^i for the different particles, and each term of this expansion corresponds to a particular scattering amplitude in $\mathcal{N} = 4$ SYM with a fixed total helicity $h_{\text{tot}} = \sum_{i=1}^n h_i$. A term containing m_i powers of η_A^i corresponds to a scattering process where the i^{th} particle has helicity $h_i = 1 - m_i/2$. Explicitly, the n -point MHV superamplitude is [32]

$$\mathcal{A}_{\text{MHV}}(1, \dots, n) = i(2\pi)^4 \frac{\delta^{(4)}(\sum_{i=1}^n \lambda_i \tilde{\lambda}_i) \delta^{(8)}(\sum_{i=1}^n \eta_i \lambda_i)}{\langle 12 \rangle \cdots \langle n1 \rangle}, \quad (2.1)$$

where, as usual, $\langle ij \rangle := \epsilon_{\alpha\beta} \lambda_i^\alpha \lambda_j^\beta$, and the two delta functions in the numerator impose momentum and supermomentum conservation, respectively. The use of superamplitudes is not only conceptually important, but also has the practical advantage of allowing for efficient ways to perform sums over internal helicities which occur at tree level and in (generalised) unitarity cuts [3, 33–39].

Let us now describe the structure of one-loop superamplitudes. The natural basis to consider in the context of dual conformal symmetry is given by the scalar box functions F_i . These are related to the scalar box integrals I_i by a kinematic prefactor [15], in the following way. We call K_1, K_2, K_3 and K_4 the external momenta at the four corners of a given box function, which are expressed as sums of momenta p_i of external particles. The momenta $K_{1\dots 4}$ can also be written in terms of the region momenta $x_{1\dots 4}$, e.g. $K_1 = x_{12}$, where $x_{ij} := x_i - x_j$ (see Figure 1). Then, up to a numerical constant, the relation between the F 's and the I 's is²

$$\begin{aligned} I_i &= -2 \frac{F_i}{\sqrt{R_i}}, \\ R_i &= (x_{13}^2 x_{24}^2)^2 - 2x_{13}^2 x_{24}^2 x_{12}^2 x_{34}^2 - 2x_{13}^2 x_{24}^2 x_{23}^2 x_{41}^2 + (x_{12}^2 x_{34}^2 - x_{23}^2 x_{41}^2)^2. \end{aligned} \quad (2.2)$$

In this paper we will not make any statement on coefficients of four-mass box functions, as these are infrared finite and trivially invariant under dual conformal symmetry. We can then simplify the expression for \sqrt{R} to

$$\sqrt{R} \rightarrow x_{13}^2 x_{24}^2 - x_{23}^2 x_{41}^2, \quad (2.3)$$

²In (2.2) we use a collective index i to denote the box function with external momenta $K_{1\dots 4}$.

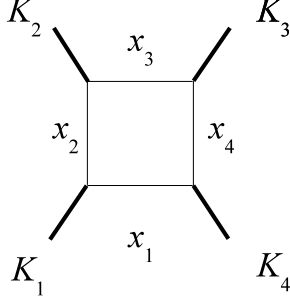


Figure 1: A generic box function. K_1, K_2, K_3 and K_4 denote the external momenta, and x_1, x_2, x_3 and x_4 the corresponding region momenta, with $K_i = x_i - x_{i+1}$, $i = 1, \dots, 4$.

valid for all box functions except four-mass ones in the case where either x_{12}^2 or x_{34}^2 vanish. Notice that, under dual conformal inversions, one has

$$\sqrt{R_i} \rightarrow \frac{\sqrt{R_i}}{x_1^2 x_2^2 x_3^2 x_4^2}. \quad (2.4)$$

We expand a generic n -point one-loop superamplitude $\mathcal{A}_n^{1\text{-loop}}$ in terms of box functions [15] as

$$\mathcal{A}_n^{1\text{-loop}} = \sum_{\{i,j,k,l\}} c(i,j,k,l) F(i,j,k,l), \quad (2.5)$$

where i, j, k, l , denote the four region momenta of the box function (as in Figure 1, with the labels 1, 2, 3, 4, replaced by i, j, k, l).

In [2] it was shown that the supercoefficients c_i have uniform covariant transformation properties under dual conformal transformations just as the corresponding tree-level amplitudes. Specifying to the different types of box functions, we get

$$\begin{aligned} \mathcal{A}_n^{1\text{-loop}} &= \sum_i c^{1\text{m}}(i) F_i^{1\text{m}} + \sum_{i,j} c^{2\text{me}}(i,j) F_{ij}^{2\text{me}} \\ &+ \sum_{i,j} c^{2\text{mh}}(i,j) F_{ij}^{2\text{mh}} + \sum_{i,j,k} c^{3\text{m}}(i,j,k) F_{ijk}^{3\text{m}} + \dots, \end{aligned} \quad (2.6)$$

where the dots stand for the four-mass box contributions. The sums in (2.6) are extended over the n one-mass, $n(n-5)/2$ two-mass easy, $n(n-5)$ two-mass hard, and $n(n-5)(n-6)/2$ three-mass box functions. Furthermore, the coefficients of the various types of box functions are related to the general box coefficients as

$$\begin{aligned} c^{1\text{m}}(i) &= c(i, i+1, i+2, i+3), & c^{2\text{me}}(i,j) &= c(i, i+1, j, j+1), \\ c^{2\text{mh}}(i,j) &= c(i, i+1, i+2, j), & c^{3\text{m}}(i,j,k) &= c(i, i+1, j, k). \end{aligned} \quad (2.7)$$

In [40] it was observed that the total number of 2me and 1m coefficients, $n(n-3)/2$, precisely equals the number of infrared consistency equations. Furthermore, it was also noticed that for n odd these equations are independent and can therefore be used to determine all the 2me and 1m coefficients in terms of the 2mh and 3m coefficients; whereas, for n even, it was checked up to $n = 29$ that one equation is actually redundant.

2.2 Anomalous one-loop amplitudes

Tree-level superamplitudes are covariant under dual superconformal symmetry, as conjectured in [1] and later proved in [2]. In order to deal with quantities which are invariant under dual conformal transformations (rather than covariant), we follow [7] and redefine the dual conformal generator K^μ to be

$$K^\mu \rightarrow \hat{K}^\mu := K^\mu + \sum_{i=1}^n x_i^\mu . \quad (2.8)$$

Then \hat{K}^μ annihilates any tree-level superamplitude.

At one loop, amplitudes are no longer covariant under four-dimensional conformal transformations as they are at tree level because of the presence of infrared divergences. Instead, according to the conjecture of [1], they are expected to have the following anomaly at one loop under a dual conformal transformation:

$$\hat{K}^\mu \mathcal{A}_n^{1\text{-loop}} = 4\epsilon \mathcal{A}_n^{\text{tree}} \sum_{i=1}^n x_{i+1}^\mu x_{i+2}^2 J(x_{i+2}^2) , \quad (2.9)$$

where the one-mass triangle $J(a)$ is defined by

$$J(a) = \frac{r_\Gamma}{\epsilon^2} (-a)^{-\epsilon-1} , \quad (2.10)$$

and $r_\Gamma := \Gamma(1+\epsilon)\Gamma^2(1-\epsilon)/\Gamma(1-2\epsilon)$.

We then apply \hat{K}^μ defined in (2.8) to both sides of (2.5) to determine the consequences of dual conformal symmetry. Acting upon the left hand side we get the dual conformal anomaly in (2.9). Acting upon the right hand side, we get $\sum_{\{i,j,k,l\}} c(i,j,k,l) K^\mu F(i,j,k,l)$ since $\hat{K}c = 0$, which is the statement that all box coefficients are covariant under dual conformal symmetry, hence

$$\hat{K}^\mu \mathcal{A}^{1\text{-loop}} = \sum_{\{i,j,k,l\}} c(i,j,k,l) K^\mu F(i,j,k,l) . \quad (2.11)$$

It is therefore clear that the anomaly stems entirely from the (anomalous) box functions. Our next task will then consist in calculating explicitly the dual conformal anomalies of all the one-loop box functions.

A comment is in order here. The statement of the anomalous conformal invariance (2.9) is not phrased in the same way as that of [1] but the two statements can easily be shown to be equivalent. In [1], dual superconformal invariance was used to argue that any superamplitude can be written as the MHV superamplitude, which captures the anomaly, times a dual superconformal invariant factor \mathcal{R} , i.e.

$$\mathcal{A} = \mathcal{A}_{\text{MHV}} \mathcal{R} . \quad (2.12)$$

Expanding this out in the coupling constant we obtain

$$\mathcal{A}^{\text{tree}} = \mathcal{A}_{\text{MHV}}^{\text{tree}} \mathcal{R}^{\text{tree}} , \quad (2.13)$$

$$\mathcal{A}^{1\text{-loop}} = \mathcal{A}_{\text{MHV}}^{\text{tree}} \mathcal{R}^{1\text{-loop}} + \mathcal{A}_{\text{MHV}}^{1\text{-loop}} \mathcal{R}^{\text{tree}} . \quad (2.14)$$

The first equation is simply the statement that all tree-level amplitudes are covariant under dual conformal symmetry which was proved in [2]. As for the second equation, by applying a conformal transformation to both sides of (2.14) and using the anomalous transformation of the one-loop MHV amplitude [11], which follows from the Wilson loop/amplitude duality,

$$\hat{K}^\mu \mathcal{A}_{\text{MHV}}^{1\text{-loop}} = 4\epsilon \mathcal{A}_{\text{MHV}}^{\text{tree}} \sum_{i=1}^n x_{i+1}^\mu x_{ii+2}^2 J(x_{ii+2}^2) , \quad (2.15)$$

we obtain directly (2.9).

2.3 Anomalies of one-loop box functions

Here we derive the anomaly of generic box functions under dual conformal transformations, which we will then use to study (2.11). As mentioned earlier, four-mass box functions are invariant under dual conformal symmetry [21], hence we will not need to consider them.

Consider a generic one-loop box function,

$$F = -i(4\pi)^{2-\epsilon} \int \frac{d^D x_5}{(2\pi)^D} \frac{(-\sqrt{R}/2)}{x_{51}^2 x_{52}^2 x_{53}^2 x_{54}^2} , \quad (2.16)$$

where \sqrt{R} is defined in (2.4) and $D = 4 - 2\epsilon$. We perform a dual conformal transformation on the region momenta of the box function,

$$x_i^\mu \rightarrow x_i'^\mu := \frac{x_i^\mu + b^\mu x_i^2}{1 + 2(bx_i) + x_i^2 b^2} , \quad (2.17)$$

where $i = 1, \dots, 4$. This transformation can be compensated by performing an identical one on the integration variable x_5 . Doing so, one obtains³

$$F(x'_1, x'_2, x'_3, x'_4) = -i(4\pi)^{2-\epsilon} \int \frac{d^D x_5}{(2\pi)^D} |J| \frac{(-\sqrt{R}/2)}{x_{51}^2 x_{52}^2 x_{53}^2 x_{54}^2} (1 + 2(bx_5) + b^2 x_5^2)^4, \quad (2.18)$$

where

$$|J| := \det_{\mu,\nu} \left(\frac{\partial x_5'^\mu}{\partial x_5'^\nu} \right) = 1 - 2D(bx_5) + \mathcal{O}(b^2). \quad (2.19)$$

x'_5 is given by an expression identical to (2.17), and we have used $\eta_{\mu\nu}\eta^{\mu\nu} = D$. We conclude that, under a dual conformal transformation, the variation of a generic box function is

$$\delta_b F = 2(4 - D) \left[-i(4\pi)^{2-\epsilon} \right] \int \frac{d^D x_5}{(2\pi)^D} \frac{(-\sqrt{R}/2)}{x_{51}^2 x_{52}^2 x_{53}^2 x_{54}^2} (bx_5). \quad (2.20)$$

The integral appearing on the right hand side of (2.20) is a linear box, which can be evaluated straightforwardly with a Passarino-Veltman (PV) reduction [41]. Crucially we have $4 - D = 2\epsilon$, hence it will be enough to pick the poles in inverse powers of ϵ in the PV reduction appearing on the right hand side of (2.20), which greatly facilitates our task.

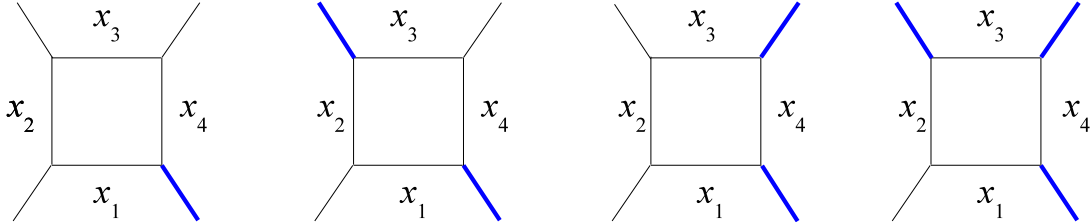


Figure 2: *The 1m, 2me, 2mh, 3m box functions whose dual conformal anomalies are calculated in (2.21). Massive legs are depicted in bold blue.*

We have performed the relevant PV reductions for the linear boxes appearing in (2.20), which in turn give the anomalous action of dual conformal generators on each type of box functions. Our results for the boxes represented in Figure 2 are the

³Recall that differences of dual momenta transform covariantly, $(x - y)^2 \rightarrow (x - y)^2 / [(1 + 2(bx) + 2b^2 x^2)(1 + 2(by) + 2b^2 y^2)]$.

following:

$$\begin{aligned}
K^\mu F^{0m} &= 2\epsilon \left[(x_1 + x_3)^\mu x_{24}^2 J(x_{24}^2) + (x_2 + x_4)^\mu x_{13}^2 J(x_{13}^2) \right] , \\
K^\mu F_1^{1m} &= -2\epsilon \left\{ -x_1^\mu x_{24}^2 J(x_{24}^2, x_{41}^2) + x_2^\mu \left[x_{41}^2 J(x_{24}^2, x_{41}^2) - x_{13}^2 J(x_{13}^2) \right] \right. \\
&\quad \left. + x_3^\mu \left[x_{41}^2 J(x_{13}^2, x_{41}^2) - x_{24}^2 J(x_{24}^2) \right] - x_4^\mu x_{13}^2 J(x_{13}^2, x_{41}^2) \right\} , \\
K^\mu F_{13}^{2me} &= -2\epsilon \left\{ x_1^\mu \left[-x_{24}^2 J(x_{24}^2, x_{41}^2) + x_{23}^2 J(x_{13}^2, x_{23}^2) \right] + x_2^\mu \left[-x_{13}^2 J(x_{13}^2, x_{23}^2) + x_{41}^2 J(x_{24}^2, x_{41}^2) \right] \right. \\
&\quad \left. + x_3^\mu \left[-x_{24}^2 J(x_{24}^2, x_{23}^2) + x_{41}^2 J(x_{13}^2, x_{41}^2) \right] + x_4^\mu \left[-x_{13}^2 J(x_{13}^2, x_{41}^2) + x_{23}^2 J(x_{24}^2, x_{23}^2) \right] \right\} , \\
K^\mu F_{14}^{2mh} &= -2\epsilon \left\{ -x_1^\mu x_{24}^2 J(x_{24}^2, x_{41}^2) \right. \\
&\quad \left. + x_2^\mu \left[x_{41}^2 J(x_{24}^2, x_{41}^2) - x_{13}^2 J(x_{13}^2) + x_{34}^2 J(x_{24}^2, x_{34}^2) \right] \right. \\
&\quad \left. - x_3^\mu x_{24}^2 J(x_{24}^2, x_{34}^2) \right\} , \\
K^\mu F_{134}^{3m} &= -2\epsilon \left\{ x_1^\mu \left[x_{23}^2 J(x_{23}^2, x_{13}^2) - x_{24}^2 J(x_{24}^2, x_{41}^2) \right] + x_2^\mu \left[x_{41}^2 J(x_{41}^2, x_{24}^2) - x_{13}^2 J(x_{13}^2, x_{23}^2) \right] \right\} .
\end{aligned} \tag{2.21}$$

The two-mass triangle $J(a, b)$ is defined by

$$J(a, b) := \frac{r_\Gamma}{\epsilon^2} \frac{(-a)^{-\epsilon} - (-b)^{-\epsilon}}{(-a) - (-b)} . \tag{2.22}$$

Notice also that $J(a, b) \rightarrow J(a)$ as $b \rightarrow 0$.

Below we also list the corresponding infrared divergent terms for the same integral box functions:

$$\begin{aligned}
F^{0m}|_{\text{IR}} &= -\frac{r_\Gamma}{\epsilon^2} \left[(-x_{13}^2)^{-\epsilon} + (-x_{24}^2)^{-\epsilon} \right] , \\
F_1^{1m}|_{\text{IR}} &= -\frac{r_\Gamma}{\epsilon^2} \left[(-x_{13}^2)^{-\epsilon} + (-x_{24}^2)^{-\epsilon} - (-x_{41}^2)^{-\epsilon} \right] , \\
F_{13}^{2me}|_{\text{IR}} &= -\frac{r_\Gamma}{\epsilon^2} \left[(-x_{13}^2)^{-\epsilon} + (-x_{24}^2)^{-\epsilon} - (-x_{23}^2)^{-\epsilon} - (-x_{41}^2)^{-\epsilon} \right] , \\
F_{14}^{2mh}|_{\text{IR}} &= -\frac{r_\Gamma}{\epsilon^2} \left[\frac{1}{2}(-x_{13}^2)^{-\epsilon} + (-x_{24}^2)^{-\epsilon} - \frac{1}{2}(-x_{34}^2)^{-\epsilon} - \frac{1}{2}(-x_{41}^2)^{-\epsilon} \right] , \\
F_{134}^{3m}|_{\text{IR}} &= -\frac{r_\Gamma}{\epsilon^2} \left[\frac{1}{2}(-x_{13}^2)^{-\epsilon} + \frac{1}{2}(-x_{24}^2)^{-\epsilon} - \frac{1}{2}(-x_{23}^2)^{-\epsilon} - \frac{1}{2}(-x_{41}^2)^{-\epsilon} \right] .
\end{aligned} \tag{2.23}$$

A comparison of the infrared divergent terms of the boxes (2.23) with the conformal variations (2.21) of the same integral functions shows that the former quantities can be obtained from the latter by simply replacing every occurrence of $\epsilon x_i^\mu x_{jk}^2 J(x_{jk}^2, x_{ik}^2)$ in (2.21) with $-r_\Gamma (-x_{jk}^2)^{-\epsilon} / (4\epsilon^2) \equiv x_{jk}^2 J(x_{jk}^2) / 4$. We will make use of this observation later on to show that relations of the box coefficients which arise from considering the infrared divergences of the amplitude are all implied by the anomalous dual conformal symmetry.

3 Conformal constraints on scattering amplitudes

In this section we examine in more detail the constraints imposed by dual conformal invariance on the coefficients of the one-loop superamplitudes. Anticipating our story a little, in the following we will find that:

1. The $n(n-3)/2$ infrared consistency equations are always implied by dual conformal invariance, hence they do not provide new relations;
2. The number of independent conformal constraints we find is equal to $n(n-4)$, regardless of n being even or odd.

Thus, conformal equations imply $n(n-4) - n(n-3)/2 = n(n-5)/2$ new constraints on top of the infrared consistency relations. We will see how the conformal equations allow to eliminate n one-mass box function coefficients, the $n(n-5)/2$ two-mass easy box coefficients, and $n(n-5)/2$ antisymmetric combinations, to be specified later, of the two-mass hard coefficients. As a result, one is left with all the three-mass coefficients and a symmetric combination of two-mass hard coefficients which are the real unknowns of the problems (and can be calculated using quadruple cuts).

3.1 Generic n -point superamplitudes

We now move on to derive the constraints on the box coefficients from dual conformal invariance for general n -point amplitudes. To illustrate our results in concrete examples, we will then briefly consider the cases $n = 4, 5, 6$.

We start our analysis from (2.11), and we use the results (2.21) for the anomalies of the box functions derived in (2.21). Examining carefully the variation of the boxes under dual conformal transformations (2.21), we see that all terms will be either of the form

$$x_i^\mu x_{i-1k}^2 J(x_{ik}^2, x_{i-1k}^2) , \quad \text{or} \quad x_{i-1}^\mu x_{ik}^2 J(x_{ik}^2, x_{i-1k}^2) , \quad (3.1)$$

for some i, k . These terms appear in the variations of the box functions $F(i, k, j, i-1)$ and $F(i, j, k, i-1)$ only, for all allowed values of j . Examining further the sign with

which these terms appear we find that

$$\begin{aligned}
\hat{K}^\mu \mathcal{A}_n^{1\text{-loop}} &= \sum_{\{i,j,k,l\}} c(i,j,k,l) K^\mu F(i,j,k,l) \\
&= -2\epsilon \sum_{i=1}^n \sum_{k=i+2}^{i+n-3} \mathcal{E}(i,k) \left[x_{i-1}^\mu x_{ik}^2 J(x_{ik}^2, x_{i-1k}^2) - x_i^\mu x_{i-1k}^2 J(x_{ik}^2, x_{i-1k}^2) \right] \\
&\quad - 2\epsilon \sum_{i=1}^n \left[\sum_{j=i+1}^{i+n-3} -c(i,j,i-2,i-1) \right] x_{i-1}^\mu x_{ii-2}^2 J(x_{ii-2}^2) , \tag{3.2}
\end{aligned}$$

where we have defined

$$\mathcal{E}(i,k) := \sum_{j=k+1}^{i+n-2} c(i,k,j,i-1) - \sum_{j=i+1}^{k-1} c(i,j,k,i-1) , \tag{3.3}$$

and the formula (3.3) is only valid for $i < k$; if $i > k$, then the variable k appearing in the summation ranges of (3.3) has to be replaced by $k+n$. Note that in (3.2) all one-mass triangles are confined to the last line. It is also important to mention that the second line in (3.2) is of order ϵ^{-1} , while the first line is of order ϵ^0 .

Let us now discuss consequences of (3.2) assuming the validity of (2.9) up to and including terms of order $\mathcal{O}(\epsilon^0)$.

First, we observe that the term in square brackets in the last line of (3.2) is nothing but the particular combination of two-mass hard and one-mass box coefficients discovered in [29]⁴

$$\sum_{j=i+1}^{i+n-3} c(i-2,i-1,i,j) = 2 \mathcal{A}_n^{\text{tree}}, \quad i = 1, \dots, n . \tag{3.4}$$

We find it interesting that the combination of infrared consistency conditions introduced in [29] and derived in [38] emerges naturally from our conformal equations. Hence, the last line of (3.2) captures entirely the dual conformal anomaly (2.9) of [1].

Second, by comparing (3.2) and (2.9) we find that the second line of (3.2) must vanish. Since each coefficient $\mathcal{E}(i,k)$ is multiplied by an independent combination of two-mass triangles they must vanish individually, leading to the following set of equations:

$$\mathcal{E}(i,k) = 0 , \quad i = 1, \dots, n , \quad k = i+2, \dots, i+n-3 . \tag{3.5}$$

⁴Incidentally, this is the same combination that appears in the BCF recursion relation [42].

These equations are among the most important new results of this paper, because they give new constraints on the box supercoefficients arising from dual conformal invariance. In Section 4 we will discuss in more detail consequences of these additional conformal equations, where we will also prove that these lead to $n(n-4)$ independent linear relations among the one-loop supercoefficients. Notice that assuming the conformal anomaly equation (2.9) we also have found an alternative proof of (3.4).

For later convenience we extend the definition of $\mathcal{E}(i, k)$ to include also $\mathcal{E}(i, i-2)$ and $\mathcal{E}(i-1, i)$,

$$\mathcal{E}(i, i-2) = -\mathcal{E}(i-1, i) := - \sum_{j=i+1}^{i+n-3} c(i-2, i-1, i, j) , \quad (3.6)$$

which allows us to rewrite (3.4) as

$$\mathcal{E}(i, i-2) = -\mathcal{E}(i-1, i) = -2 \mathcal{A}_n^{\text{tree}} , \quad i = 1, \dots, n . \quad (3.7)$$

3.2 Examples of applications of the conformal equations

In order to make more concrete the discussion presented in the previous section, we would now like to apply (3.7) and (3.5) to the case of four-, five- and six-point superamplitudes.

3.2.1 Four- and five-point amplitudes

The four- and five-point cases are simple enough that we can address them directly, without passing through the n -point conformal equations derived in (3.5), (3.7).

At four points, the only box function that can appear is the massless box. The one-loop superamplitude can then be written as

$$\mathcal{A}_4^{1\text{-loop}} = c^{0\text{m}} F^{0\text{m}} . \quad (3.8)$$

The action of a special conformal transformation on the superamplitude is expected to produce the anomaly (2.9). Explicitly, we expect to find

$$\hat{K}^\mu \mathcal{A}_4^{1\text{-loop}} = 4\epsilon \mathcal{A}_4^{\text{tree}} \left[(x_2^\mu + x_4^\mu) x_{13}^2 J(x_{13}^2) + (x_1^\mu + x_3^\mu) x_{24}^2 J(x_{24}^2) \right] . \quad (3.9)$$

By acting directly on (3.8), we obtain

$$\hat{K}^\mu \mathcal{A}_4^{1\text{-loop}} = c^{0\text{m}} K^\mu F^{0\text{m}} . \quad (3.10)$$

In writing (3.10), we have used the invariance of the box coefficients under the action of \hat{K}_μ [1, 2].

The conformal transformation of the zero-mass box function is given in (2.21), and inserting this in (3.10) we find that the one-loop four-point amplitude is dual conformally invariant as long as $c^{0m} = 2 \mathcal{A}_4^{\text{tree}}$, i.e.

$$\mathcal{A}_4^{1\text{-loop}} = 2 \mathcal{A}_4^{\text{tree}} F^{0m} . \quad (3.11)$$

At five points, only one-mass box functions can appear, therefore the generic one-loop superamplitude can be written as

$$\mathcal{A}_5^{1\text{-loop}} = \sum_{i=1}^5 c_i^{1m} F_i^{1m} . \quad (3.12)$$

Applying \hat{K}^μ to the left hand side of (3.12) and using the expression for the anomaly in (2.9), one expects to find

$$\hat{K}^\mu \mathcal{A}_5^{1\text{-loop}} = 4\epsilon \mathcal{A}_5^{\text{tree}} x_1^\mu x_{25}^2 J(x_{25}^2) + \text{cyclic} . \quad (3.13)$$

Here ‘+ cyclic’ indicates the addition of four terms obtained by rewriting the labels as $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 1$. Now applying \hat{K}^μ to the right hand side of (3.12) gives

$$\begin{aligned} \sum_{i=1}^5 c_i^{1m} \hat{K}^\mu F_i^{1m} = & -2\epsilon c_1^{1m} \left\{ -x_1^\mu x_{24}^2 J(x_{24}^2, x_{41}^2) + x_2^\mu \left[x_{41}^2 J(x_{24}^2, x_{41}^2) - x_{13}^2 J(x_{13}^2) \right] \right. \\ & \left. + x_3^\mu \left[x_{41}^2 J(x_{13}^2, x_{41}^2) - x_{24}^2 J(x_{24}^2) \right] - x_4^\mu x_{13}^2 J(x_{13}^2, x_{41}^2) \right\} + \text{cyclic} , \end{aligned} \quad (3.14)$$

or, after rearranging terms,

$$\begin{aligned} \sum_{i=1}^5 c_i^{1m} \hat{K}^\mu F_i^{1m} = & -2\epsilon x_1^\mu \left\{ (c_4^{1m} - c_1^{1m}) x_{24}^2 J(x_{24}^2, x_{41}^2) + (c_5^{1m} - c_3^{1m}) x_{35}^2 J(x_{13}^2, x_{35}^2) \right. \\ & \left. - (c_4^{1m} + c_5^{1m}) x_{25}^2 J(x_{25}^2) \right\} + \text{cyclic} . \end{aligned} \quad (3.15)$$

Equation (3.14) comes directly from (2.21), whereas in (3.15) we have collected all terms proportional to x_1^μ .

On comparing (3.15) with (3.13) we learn that, for dual conformal invariance to hold, the first line of (3.15) must vanish, implying that all the one-mass coefficients are equal; furthermore, from the second line of (3.15), we also obtain

$$c_i^{1m} = \mathcal{A}_5^{\text{tree}} , \quad i = 1, \dots, 5 . \quad (3.16)$$

We thus find that at five points,

$$\mathcal{A}_5^{1\text{-loop}} = \mathcal{A}_5^{\text{tree}} \sum_{i=1}^5 F_i^{1\text{m}} . \quad (3.17)$$

Note that since we consider the full superamplitude, the above results (3.11) and (3.17) apply to all four- and five-point superamplitudes in $\mathcal{N}=4$ SYM. Of course, all four-point amplitudes are MHV, and all five-point amplitudes are either MHV or anti-MHV, and the above results (3.11) and (3.17) are well-known. We also notice that these results can also be obtained by just imposing the known infrared consistency conditions on the one-loop superamplitudes.

3.2.2 Six-point amplitudes

At six points, there are a total of six possible two-mass hard coefficients, $c^{2\text{mh}}(i, i+4)$, three two-mass easy coefficients $c^{2\text{me}}(i, i+3)$, six one-mass coefficients $c^{1\text{m}}(i)$ and no three-mass coefficients.

We will now use the conformal equations (3.5), (3.7) to find relations among these coefficients. At the six-point level, these equations become

$$\mathcal{E}(i, i+1) \equiv c^{1\text{m}}(i-1) + c^{2\text{mh}}(i-1, i+3) + c^{1\text{m}}(i-2) = 2 \mathcal{A}_6^{\text{tree}} , \quad (3.18)$$

$$\mathcal{E}(i, i+2) \equiv c^{2\text{mh}}(i-2, i+2) - c^{1\text{m}}(i-1) + c^{2\text{me}}(i-1, i+2) = 0 , \quad (3.19)$$

$$\mathcal{E}(i, i+3) \equiv c^{1\text{m}}(i+3) - c^{2\text{mh}}(i-1, i+3) - c^{2\text{me}}(i-1, i+2) = 0 . \quad (3.20)$$

The second and third equations imply that there are in fact only three independent two-mass hard coefficients, since

$$\begin{aligned} c^{2\text{mh}}(4, 1) &= c^{2\text{mh}}(1, 5) , \\ c^{2\text{mh}}(5, 2) &= c^{2\text{mh}}(2, 6) , \\ c^{2\text{mh}}(6, 3) &= c^{2\text{mh}}(3, 1) . \end{aligned} \quad (3.21)$$

We can choose for instance $c^{2\text{mh}}(1, 5)$, $c^{2\text{mh}}(2, 6)$, and $c^{2\text{mh}}(3, 1)$.

Equations (3.19) and (3.20) can also be used to show that all two-mass easy coefficients are equal and so there is only one independent two-mass easy coefficient (e.g. $c^{2\text{me}}(1, 4)$). As for the one-mass coefficients, these are determined by the remaining independent two-mass coefficients:

$$\begin{aligned} c^{1\text{m}}(1) &= c^{1\text{m}}(4) = c^{2\text{mh}}(3, 1) + c^{2\text{me}}(1, 4) , \\ c^{1\text{m}}(2) &= c^{1\text{m}}(5) = c^{2\text{mh}}(1, 5) + c^{2\text{me}}(1, 4) , \\ c^{1\text{m}}(3) &= c^{1\text{m}}(6) = c^{2\text{mh}}(2, 6) + c^{2\text{me}}(1, 4) . \end{aligned} \quad (3.22)$$

Finally the first equation (3.18) gives one of the remaining four independent coefficients in terms of the tree-level superamplitude $\mathcal{A}_6^{\text{tree}}$. For example,

$$c^{2\text{me}}(1, 4) = \mathcal{A}_6^{\text{tree}} - \frac{1}{2} [c^{2\text{mh}}(1, 5) + c^{2\text{mh}}(2, 6) + c^{2\text{mh}}(3, 1)] . \quad (3.23)$$

Notice that these equations hold for arbitrary six-point superamplitudes, i.e. they apply to both MHV and NMHV helicity assignments. The MHV case is instantly obtained from the previous relations by further setting $c_{\text{MHV}}^{2\text{mh}} = 0$, so that $c_{\text{MHV}}^{2\text{me}} = \mathcal{A}_{\text{MHV}}^{\text{tree}}$. For the NMHV case, consider for instance the gluonic amplitudes calculated in [43]. From [43] we infer that the sum of coefficients in square brackets in (3.23) is equal to $2\mathcal{A}^{\text{tree}}$, and hence all the two-mass easy coefficients are zero, in agreement with [43].

The general MHV superamplitude is discussed further in Section 5.1, while the n -point NMHV superamplitudes are discussed in Section 5.2.

4 Analysis of the conformal equations

In this section we discuss the conformal equations derived earlier in (3.7) and (3.5). Firstly, we will show that the conformal equations imply the infrared consistency conditions at one loop. We will then move on to analyse what are the new constraints on supercoefficients arising from the conformal equation in addition to the infrared ones.

4.1 Comparison with infrared consistency conditions

Here we compare the above equations from dual conformal invariance with the infrared consistency conditions for one-loop scattering amplitudes. We will show that the infrared equations can be derived in a neat way from the conformal equations. As anticipated earlier, in addition to these, the conformal equations also contain new relations which further constrain the one-loop supercoefficients. In the last part of this section, we will present these relations explicitly.

We start off by recalling that the infrared consistency conditions arise from the relation

$$\mathcal{A}_n^{1\text{-loop}}|_{\text{IR}} = -r_{\Gamma} \mathcal{A}_n^{\text{tree}} \sum_{i=1}^n \frac{(-x_{i+2}^2)^{-\epsilon}}{\epsilon^2} . \quad (4.1)$$

Now we wish to rewrite this equation as

$$\mathcal{A}_n^{1\text{-loop}}|_{\text{IR}} := \frac{r_\Gamma}{2\epsilon^2} \sum_{i=1}^n \sum_{k=i+2}^{i+n-2} \mathcal{I}(i, k) (-x_{ik}^2)^{-\epsilon}, \quad (4.2)$$

where obviously $\mathcal{I}(i, k) = \mathcal{I}(k, i)$. By comparing (4.1) with (4.2) we obtain the following infrared consistency conditions:

$$\mathcal{I}(i, i+2) = \mathcal{I}(i+2, i) = -2\mathcal{A}_n^{\text{tree}}, \quad (4.3)$$

$$\mathcal{I}(i, k) = 0, \quad k = i+3, \dots, i-3. \quad (4.4)$$

Now we would like to re-express $\mathcal{I}(i, i+2)$ and $\mathcal{I}(i, k)$ in terms of the particular combinations of coefficients $\mathcal{E}(i, k)$ introduced earlier. In order to do so, we first observe that the conformal variation of the one-loop amplitude (3.2) can be written more compactly as

$$\hat{K}^\mu \mathcal{A}_n^{1\text{-loop}} = -2\epsilon \sum_{i=1}^n \sum_{k=i+2}^{i+n-2} \mathcal{E}(i, k) [x_{i-1}^\mu x_{ik}^2 J(x_{ik}^2, x_{i-1k}^2) - x_i^\mu x_{i-1k}^2 J(x_{ik}^2, x_{i-1k}^2)], \quad (4.5)$$

where $\mathcal{E}(i, i-2)$ is defined in (3.6).

Next, we use the mapping between the conformal variations of the boxes with their infrared divergent parts discussed below (2.23). According to this mapping, in order to obtain the infrared divergent terms from the conformal anomalous terms, every occurrence in the latter of $\epsilon x_i^\mu x_{jk}^2 J(x_{jk}^2, x_{ik}^2)$ is mapped to a term $-r_\Gamma (-x_{jk}^2)^{-\epsilon}/(4\epsilon^2)$ in the former. Applying this map to (4.5), we arrive at the following structure for the infrared divergent terms of the amplitude:

$$\mathcal{A}_n^{1\text{-loop}}|_{\text{IR}} = \frac{r_\Gamma}{2\epsilon^2} \sum_{i=1}^n \sum_{k=i+2}^{i+n-2} \mathcal{E}(i, k) [(-x_{ik}^2)^{-\epsilon} - (-x_{i-1k}^2)^{-\epsilon}]. \quad (4.6)$$

Symmetrising the right hand side of (4.6) in i and k , we arrive at

$$\frac{r_\Gamma}{4\epsilon^2} \sum_{i=1}^n \sum_{k=i+2}^{i+n-2} \left\{ \mathcal{E}(i, k) [(-x_{ik}^2)^{-\epsilon} - (-x_{i-1k}^2)^{-\epsilon}] + \mathcal{E}(k, i) [(-x_{ik}^2)^{-\epsilon} - (-x_{ik-1}^2)^{-\epsilon}] \right\}, \quad (4.7)$$

and finally we manipulate this expression by collecting together all terms proportional to $(-x_{ik}^2)^{-\epsilon}$ by shifting indices and changing summation boundaries appropriately.

Thus we find

$$\begin{aligned}\mathcal{A}_n^{1\text{-loop}}|_{\text{IR}} &= \frac{r_\Gamma}{4\epsilon^2} \sum_{i=1}^n \sum_{k=i+3}^{i+n-3} [\mathcal{E}(i, k) + \mathcal{E}(k, i) - \mathcal{E}(i+1, k) - \mathcal{E}(k+1, i)] (-x_{ik}^2)^{-\epsilon} \\ &+ \frac{r_\Gamma}{2\epsilon^2} \sum_{i=1}^n [\mathcal{E}(i, i+2) + \mathcal{E}(i+2, i) - \mathcal{E}(i+3, i)] (-x_{i,i+2}^2)^{-\epsilon} .\end{aligned}\quad (4.8)$$

Note that special care has to be taken at the boundaries of the summation for coefficients of $(-x_{i,i+2}^2)^{-\epsilon}$ which leads to the second line in (4.8).

Comparing (4.8) to (4.2) we immediately get

$$\mathcal{I}(i, i+2) = \mathcal{E}(i, i+2) + \mathcal{E}(i+2, i) - \mathcal{E}(i+3, i) , \quad (4.9)$$

$$\mathcal{I}(i, k) = \mathcal{E}(i, k) + \mathcal{E}(k, i) - \mathcal{E}(i+1, k) - \mathcal{E}(k+1, i) , \quad k = i+3, \dots, i+n-3 .$$

Finally, by plugging the conformal equations (3.5) and (3.7) into (4.9) one obtains the infrared consistency conditions (4.3) and (4.4).

We have mentioned earlier that the particular combination of infrared equations introduced in [29] and proven in [38] emerges naturally from our conformal equations, namely it appears in (3.7). Here we would like to add that this combination arises from considering the sum $\sum_{k=i+2}^{i+n-2} \mathcal{I}(i, k)$ of infrared equations. Indeed, using the $2n$ algebraic identities

$$\sum_{k=i+1}^{i+n-2} \mathcal{E}(i, k) = 0 , \quad (4.10)$$

$$\mathcal{E}(i, i+1) = -\mathcal{E}(i+1, i+n-1) , \quad (4.11)$$

which can easily be checked by using the definition of $\mathcal{E}(i, k)$ given in (3.3), one obtains

$$\sum_{k=i+2}^{i+n-2} \mathcal{I}(i, k) = 2\mathcal{E}(i+1, i-1) = -4\mathcal{A}_{\text{tree}} , \quad (4.12)$$

which is the combination of [29].

Finally, one may want to disentangle the genuinely new conformal equations from the infrared consistency conditions. This can be achieved in the following way.

Firstly, we consider the following linear combinations of the dual conformal equa-

tions (3.7) and (3.5):

$$\begin{aligned}
\mathcal{F}(i, k) &\equiv \sum_{l=i+1}^{k-1} \mathcal{E}(i, l) \equiv \sum_{l=i+1}^{k-1} \sum_{m=k}^{i+n-2} c(i, l, m, i-1) \\
&\equiv c^{1m}(i-2) + c^{2me}(i-1, k-1) + (2mh, 3m \text{ coefficients}) \\
&= 2 \mathcal{A}_n^{\text{tree}},
\end{aligned} \tag{4.13}$$

where $k = i+2, \dots, i+n-2$. The last equality in (4.13) follows from the conformal equations (3.7) and (3.5).

On examination of the explicit expression in terms of box functions it is clear that

$$\mathcal{F}(i, i+2) \equiv \mathcal{F}(i+1, i-1). \tag{4.14}$$

Other than this all equations are independent, and since there are $n(n-4)$ equations, the \mathcal{F} equations are completely equivalent to the original \mathcal{E} equations.

Now we consider the antisymmetric combination $\mathcal{F}(i, k) - \mathcal{F}(k, i)$, and notice that it is independent of the two-mass easy box coefficients since $c^{2me}(i-1, k-1) = c^{2me}(k-1, i-1)$. We can further find combinations which are completely independent of two-mass easy as well as of the one-mass box coefficients. These are given by the following expression,

$$\mathcal{G}(i, k) := \mathcal{F}(i, k) - \mathcal{F}(i+1, k) - \mathcal{F}(i, k+1) + \mathcal{F}(i+1, k+1) - (i \leftrightarrow k), \tag{4.15}$$

where $k = i+3, \dots, i-3$. Since $\mathcal{G}(i, k) = -\mathcal{G}(k, i)$, and for each $i = 1, \dots, n$, the index k takes on $n-5$ values, we have $n(n-5)/2$ such equations. These are genuinely new relations from dual conformal invariance, i.e. they do not already arise from infrared consistency conditions alone.

4.2 Solution to the conformal equations

We begin by discussing how many linearly independent relations arise from the conformal equations. We have $n(n-4)$ equations from (3.5), to which we have to add the $2n$ anomaly conditions (3.7), which in total give $n(n-2)$ conformal equations. Now we have found $2n$ algebraic relations in (4.10) and (4.11). This leaves exactly $n(n-4)$ linearly independent conformal equations.⁵

An independent set of equations is given by considering the quantities $\mathcal{E}(i, k)$ for

⁵We have checked numerically up to $n = 14$ that the remaining equations are indeed independent.

$k = i + 2, \dots, i + n - 3$, which satisfy the relations

$$\sum_{i=1}^n \sum_{k=i+2}^{i+n-3} \mathcal{E}(i, k) = 0 . \quad (4.16)$$

In addition, we include a single equation giving the conformal anomaly, (3.7).

The next question we would like to address is how can we use the conformal equations to solve for particular sets of coefficients of the one-loop amplitudes. We have addressed this issue by studying with an algebraic manipulation program the system of linear equations (3.5) and (3.7), and solving for different choices of coefficients. If we define the particular (anti-)symmetric combinations of two-mass hard coefficients

$$c_a^{2\text{mh}}(i, j) := c^{2\text{mh}}(i, j) - c^{2\text{mh}}(j - 1, i + 1) , \quad (4.17)$$

$$c_s^{2\text{mh}}(i, j) := c^{2\text{mh}}(i, j) + c^{2\text{mh}}(j - 1, i + 1) , \quad (4.18)$$

then the conformal equations can be generally recast as the following $n(n - 4)$ equations

$$c^{1\text{m}}(i) = f_i^{1\text{m}}(c_s^{2\text{mh}}, c^{3\text{m}}) , \quad (4.19)$$

$$c^{2\text{me}}(i, j) = f_{ij}^{2\text{me}}(c_s^{2\text{mh}}, c^{3\text{m}}) , \quad (4.20)$$

$$c_a^{2\text{mh}}(i, j) = f_{ij}^{2\text{mh};a}(c_s^{2\text{mh}}, c^{3\text{m}}) , \quad (4.21)$$

thus enabling us to solve for the $n(n - 5)/2$ antisymmetric combination of two-mass hard coefficients, as well as for the n one-mass and $n(n - 5)/2$ two-mass easy coefficients in terms of the symmetric combination of two-mass hard coefficients, together with three-mass coefficients only. The precise functions in the equations above appear to be quite complicated but we have explicitly checked that this rearrangement is possible up to $n = 12$.

One comment is in order here. It is known that for n odd, one may use the infrared equations to solve for all one-mass and two-mass box coefficients in terms of two-mass hard and three-mass coefficients [40]. For n even, this is no longer possible since one of the infrared consistency equations can no longer be used to this purpose. In our discussion before, we find that the cases of n even and odd are not distinguished.⁶

Summarising, we have shown that, after imposing dual conformal invariance, the coefficients $c^{1\text{m}}, c^{2\text{me}}$ and $c_a^{2\text{mh}}$ are completely determined in terms of the remaining undetermined coefficients $c_s^{2\text{mh}}$ and $c^{3\text{m}}$.

⁶Another possible approach would have been to solve for the two-mass hard coefficients and the one-mass coefficients in terms of the two-mass easy and the three-mass ones. It turns out that one can do this when n is not divisible by 3.

5 The anomaly equations for MHV and NMHV superamplitudes

The anomaly equations (3.5) and (3.7) found above give relations between box coefficients of generic superamplitudes. In this section we will specialise to particular cases, namely to the MHV, anti-MHV and NMHV superamplitudes, in order to provide explicit checks of these equations.

5.1 MHV (and anti-MHV) amplitudes

MHV superamplitudes (and hence also the corresponding box supercoefficients) have a total of eight η 's. Calculating these using unitarity cuts one quickly sees that two tree-level three-point anti-MHV amplitudes are required at two of the corners and two tree-level MHV amplitudes at the other two corners. Since anti-MHV amplitudes can not be adjacent, the three-point anti-MHV amplitudes must lie at opposite corners of the box and hence only two-mass easy and one-mass box coefficients are non-vanishing.

Let us now consider (3.5). For the MHV case this becomes, after shifting i to $i+1$ for convenience,

$$\mathcal{E}(i+1, k) = c^{2\text{me}}(i, k) - c^{2\text{me}}(i, k-1) = 0, \quad (5.1)$$

with $k = i+3, \dots, i+n-2$. Considering this relation for all allowed values of k one obtains that

$$c^{1\text{m}}(i) = c^{2\text{me}}(i, k) = c^{1\text{m}}(k), \quad (5.2)$$

where in the last step we have used $c^{2\text{me}}(i, k) = c^{2\text{me}}(k, i)$. In other words, all the one-mass and two-mass easy coefficients are equal to each other. To determine their common value we use the anomaly equation (3.7), which becomes

$$\mathcal{E}(i, i-2) = -(c^{1\text{m}}(i-2) + c^{1\text{m}}(i-3)) = -2\mathcal{A}_n^{\text{tree}}, \quad (5.3)$$

from which we infer that all the supercoefficients are equal to $\mathcal{A}_n^{\text{tree}}$.

The four- and five-point cases discussed in section 3.2.1 are special cases of this since one can only have MHV or anti-MHV amplitudes in these cases.

5.2 NMHV amplitudes

The infinite sequence of one-loop NMHV gluon scattering amplitudes has been calculated in [40]. Here we will consider the NMHV superamplitudes, which have been calculated recently in [3] using supersymmetric generalised unitarity. There the coefficients are all expressed in terms of the quantities R_{rst} , defined as

$$R_{rst} := \mathcal{A}_{n,\text{MHV}}^{\text{tree}} \frac{\langle s-1 s \rangle \langle t-1 t \rangle \delta^{(4)}(\Xi_{rst})}{x_{st}^2 \langle r | x_{rt} x_{ts} | s-1 \rangle \langle r | x_{rt} x_{ts} | s \rangle \langle r | x_{rs} x_{st} | t-1 \rangle \langle r | x_{rs} x_{st} | t \rangle}, \quad (5.4)$$

where

$$\Xi_{rst} := \langle r | \left[x_{rs} x_{st} \sum_{k=t}^{r-1} |k\rangle \eta_k + x_{rt} x_{ts} \sum_{k=s}^{r-1} |k\rangle \eta_k \right]. \quad (5.5)$$

Notice that compared to [1] we have included for convenience a factor of $\mathcal{A}_{n,\text{MHV}}^{\text{tree}}$ into the definition of R_{rst} . With this definition, the R_{rst} are directly equal to the three-mass coefficients $c^{3\text{m}}(r, s, t)$ whenever r, s, t lie in the appropriate range, $r+2 < s < t-1 < r+n-2$. In fact, all box coefficients can be written in terms of R_{rst} as follows [3]:

$$c^{3\text{m}}(r, s, t) = R_{rst}, \quad (5.6)$$

$$c^{2\text{mh}}(r, t) = R_{r,r+2,t} + R_{r+1,t,r}, \quad (5.7)$$

$$c^{2\text{me}}(r, s) = \sum_{u,v=s+1}^{r+n-1} R_{r,u,v} + \sum_{u,v=r+1}^{s-1} R_{s,u,v}, \quad (5.8)$$

$$\begin{aligned} c^{1\text{m}}(r-2) &= c^{2\text{me}}(r, r-2) + R_{r-1,r+1,r-2} \\ &= \sum_{u,v=r+1}^{r+n-3} R_{r-2,u,v} + R_{r-1,r+1,r-2}. \end{aligned} \quad (5.9)$$

Note that R_{uv} is only defined for $u \geq r+2$, $v \geq u+2$ and $v \leq r+n-1$. We will define $R_{rst} = 0$ outside these bounds. Thus $\sum_{u,v=s+1}^{r+n-1} := \sum_{u=s+1}^{r+n-3} \sum_{v=u+2}^{r+n-1} R_{r,u,v}$ and $\sum_{u,v=r+1}^{r+n-3} := \sum_{u=r+1}^{r+n-5} \sum_{v=u+2}^{r+n-3} R_{r-2,u,v}$ etc.

A crucial relation between the R 's, which we will use in the following, is [1]

$$R_{r+2,s,r+1} = R_{r,r+2,s}. \quad (5.10)$$

Next, we would like to prove that all NMHV superamplitudes satisfy the conformal equations (3.5), (3.7), and are thus dual conformal invariant. Specifying these

equations to the NMHV case, and by inserting the expressions for the NMHV box supercoefficients given in (5.6)–(5.8) into the conformal equations, one obtains

$$\begin{aligned}
\mathcal{E}(i+1, i+2) &= c^{1m}(i) + c^{1m}(i-1) + \sum_{j=i+4}^{i-2} c^{2mh}(i, j) \\
&= \sum_{u,v=i+2}^{i+n-1} R_{iuv} + \sum_{u,v=i+1}^{i+n-2} R_{i-1uv} \\
&= 2 \mathcal{A}_{\text{tree}} ,
\end{aligned} \tag{5.11}$$

as well as

$$\begin{aligned}
\mathcal{E}(i+1, k) &= c^{2me}(i, k) - c^{2me}(i, k-1) + \sum_{j=k+2}^{i+n-2} c^{3m}(i, k, j) \\
&\quad - \sum_{j=i+3}^{k-2} c^{3m}(i, j, k) + c^{2mh}(i-1, k) - c^{2mh}(i, k) \\
&= \sum_{u,v=i+1}^{k-1} R_{kuv} - \sum_{u,v=i+1}^{k-2} R_{k-1uv} - \sum_{u=i+2}^{k-2} R_{iuk} \\
&= \left(\sum_{u,v=i+1}^{k-1} R_{kuv} - \sum_{u,v=i+2}^k R_{iuv} \right) - \left(\sum_{u,v=i+1}^{k-2} R_{k-1uv} - \sum_{u,v=i+2}^{k-1} R_{iuv} \right) \\
&= 0 ,
\end{aligned} \tag{5.12}$$

where $k = i+3, \dots, i+n-2$. We have also used the identity (5.10) to cancel some terms. Note that although the first equality of (5.12) is strictly true only for $k = i+4, \dots, i+n-3$ the second and third equalities come out correct for the two cases $k = i+3$ and $k = i+n-2$ also. In the sums over u, v it is always assumed that $u \leq v-2$.

Since the tree-level NMHV superamplitude is given by [1, 3, 44]

$$\mathcal{A}^{\text{tree}} = \sum_{u,v=i+2}^{i+n-1} R_{iuv} , \tag{5.13}$$

which is valid for any value of i , (5.11) is clearly true. As for the relations in (5.12), they are satisfied if the following relation holds:

$$\sum_{u,v=r+2}^s R_{ruv} = \sum_{u,v=r+1}^{s-1} R_{su v} , \quad r+5 \leq s \leq r+n-1 . \tag{5.14}$$

Equation (5.14) is a new identity for the R_{rst} , and the NMHV superamplitude is dual conformal invariant only if this new identity is true. In Appendix A we prove that this new identity is indeed true, and hence we have proved that all NMHV superamplitudes are dual conformal invariant.

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A Dual conformal invariance of all NMHV superamplitudes

In Section 5.2 we have reduced the problem of proving dual conformal invariance of the NMHV superamplitude to that of proving the new identity (5.14). In the following we give the proof of this relation.

The two-mass easy box coefficients of the one-loop NMHV superamplitude are given by the sum of two supersymmetric quadruple cut diagrams [3] represented in Figure 3. When computing these diagrams, one needs to insert the tree-level NMHV superamplitude (5.13) at the corner N in Figure 3. There are then two natural choices for the label i in (5.13), corresponding to either of the two loop momenta leaving the corner N . These two choices give two different expressions for the cut diagram. Consider for instance the first cut diagram. It turns out that both sides of the stronger identity (5.14) precisely correspond to the two different forms for this quadruple cut diagram. This observation proves the identity (5.14), and hence completes the proof that all NMHV superamplitudes are dual conformally covariant.

The identity (5.14) is very similar to a weaker identity,

$$\sum_{u,v=r+2}^s R_{ruv} + \sum_{u,v=s+2}^{r+n} R_{su v} = \sum_{u,v=s+1}^{r+n-1} R_{ruv} + \sum_{u,v=r+1}^{s-1} R_{su v} , \quad (\text{A.1})$$

where without loss of generality we restrict $r+5 \leq s \leq r+n-1$. This identity was conjectured in the bosonic case in [40], and proved in the supersymmetric case in [3]. The proof used a similar argument to that outlined above but applied to the two-mass

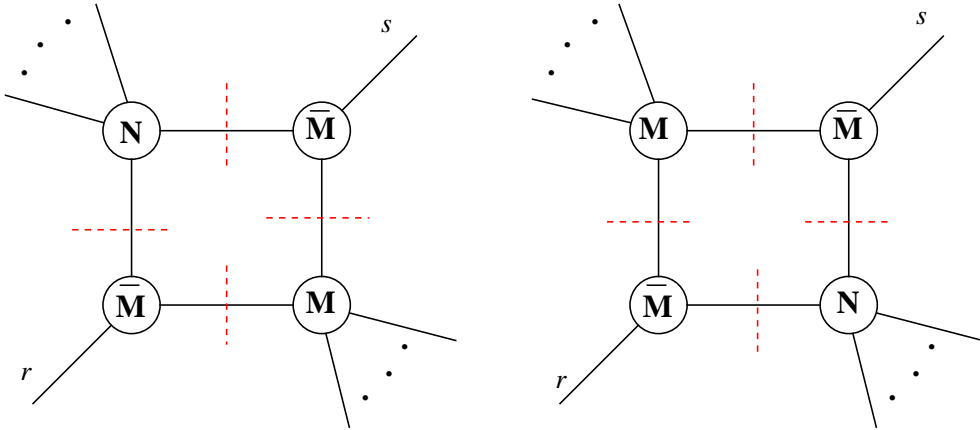


Figure 3: *The two cut diagrams which contribute to the 2me box function coefficient of the NMHV amplitude. Here in the corners M represents a tree-level MHV amplitude, \bar{M} the three-point anti-MHV amplitude and N the tree-level NMHV amplitude.*

easy box coefficient itself (i.e. the sum of the two cut diagrams). The two sides of the weaker identity (A.1) then give two different representations of the two-mass easy box coefficient $c^{2\text{me}}(r, s)$ [3].

Interestingly, it turns out that the weaker relation (A.1) is completely equivalent to the stronger one (5.14) whenever s is restricted to $r + n - 4 \leq s \leq r + n - 1$. To see this, note that the second sum on the left-hand side and the first sum on the right-hand side of (A.1) are empty for $r + n - 3 \leq s \leq r + n - 1$ (recall that $u \leq v - 2$). For the case $s = r + n - 4$, on the other hand, there is only one term in the second sum on the left-hand side and the first sum on the right-hand side of (A.1), and using the known identity $R_{r-4r-2r} = R_{rr-3r-1}$, obtained from two applications of (5.10), we again find the equivalence of (5.14) and (A.1). Therefore, (5.14) is a new identity only when $r + 5 \leq s \leq r + n - 5$, which requires $n \geq 10$.

Summarising, (A.1) is sufficient to prove dual conformal invariance of the NMHV amplitude as long as $n \leq 9$, as was done in [3]. The new identity we presented in (5.14) is required for a proof valid for arbitrary n .

We conclude this appendix by mentioning an alternative proof of (5.14). As a straightforward consequence of the form of the tree-level NMHV superamplitude (5.13), we obtain

$$\sum_{u,v=r+2}^s R_{ruv} = \sum_{u,v=r+1}^{s-1} R_{suw} . \quad (\text{A.2})$$

in the special case when the number of scattered particles is $s - r + 1$. This is

just the identity (5.14) that we wish to prove, for this restriction on the number of particles. If we now assume that this identity does not require (super-)momentum conservation⁷, then we can use exactly the same identity for an arbitrary number of external particles. In other words, we can relax the constraint on s , thus obtaining the identity (5.14) for arbitrary r, s , as required.

B Dual conformal combinations of box functions

The equations (3.5) and (3.7) can be simply adapted for a closely related purpose, namely that of finding conformally invariant linear combinations of box functions with constant coefficients. The steps needed to solve this problem are very similar to those leading to (3.5) and (3.7). There we consider a linear combination of boxes (with non-constant coefficients) and seek anomalous dual conformal invariance.

Now we require instead that the coefficients are constant, and that there is no anomaly. The resulting equations for the constant coefficients are therefore just

$$\mathcal{E}(i, k) = 0, \quad i = 1, \dots, n, \quad k = i + 1, \dots, i + n - 2. \quad (\text{B.1})$$

There are $n(n - 4)$ such equations constraining $(1/2)n(n - 5)(n - 3) + n$ coefficients, leaving $(1/2)n(n - 5)^2$ dual conformal invariant combinations. This should be compared with finite combinations of boxes. These are linear combinations of boxes whose constant coefficients satisfy

$$\mathcal{I}(i, k) = 0, \quad k = i + 2, \dots, i + n - 2. \quad (\text{B.2})$$

giving $n(n - 3)/2$ independent equations for $(1/2)n(n - 5)(n - 3) + n$ coefficients. There are therefore $n(n - 5)(n - 4)/2$ finite combinations of boxes.

For example at six points there are 6 finite combinations and 3 of these are also dual conformally invariant. This differs from a statement in [3] that at six points all finite combinations of boxes are conformally invariant. These six finite combinations are given by

$$v_i = -2F_i^{1m} - F_{i-1\,i+2}^{2me} - F_{i\,i+3}^{2me} - F_{i+1\,i+4}^{2me} + 2F_{i\,i+4}^{2mh} + 2F_{i+1\,i-1}^{2mh}, \quad (\text{B.3})$$

for $i = 1, \dots, 6$. The 3 dual conformally covariant combinations (which were given

⁷This is claimed in [3] as long as we take the identity precisely as written, and do not use (super)-momentum conservation to write, for example, $R_{suv} = R_{s-n\,uv}$ etc.

in [3] and denoted there as v_{146} , v_{251} and v_{362}) are

$$\begin{aligned} v_{146} &= \frac{1}{4} (v_0 + v_1 - v_2 + v_3 + v_4 - v_5) , \\ v_{251} &= \frac{1}{4} (-v_0 + v_1 + v_2 - v_3 + v_4 + v_5) , \\ v_{362} &= \frac{1}{4} (v_0 - v_1 + v_2 + v_3 - v_4 + v_5) . \end{aligned} \tag{B.4}$$

At seven points there are 21 finite combinations of boxes and 14 of these are conformally invariant. This agrees with the findings of [3].

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